Research Paper

FIXED POINT THEOREMS IN FUZZY 2-METRIC SPACE WITH IMPLICIT MAP

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In this paper, we prove some common fixed point theorem for four and six mappings on fuzzy 2-metric spaces by using implicit relations. Our result is an extension of existing results in fuzzy 2-metric spaces.

**Keywords:** Fuzzy 2-metric spaces, Fixed point, Contractive mapping, Implicit relations

INTRODUCTION


PRELIMINARIES

**Definition 2.1:** A triangular norm (t norm)* is a binary operation on the unit interval [0; 1] such that for all a; b; c; d ∈ [0; 1] the following conditions are satisfied:

1. a * 1 = a
2. a * b = b * a
3. a * b ≤ c * d; whenever a ≤ c and b ≤ d
4. a * (b * c) = (a * b) * c

**Definition 2.2:** The 3-tuple (X;M; *) is called a fuzzy 2-metric space if X is an arbitrary set, * is a continuous t norm and M is a fuzzy set in X^3 × X.
[0; \infty) satisfying the following conditions: for all \( x; y; z; u \in X \) and \( t_1; t_2; t_3 > 0 \)
1. \( M(x; y; z; 0) = 0 \)
2. \( M(x; y; z; t) = 1; t > 0 \) and when at least two of the three points are equal.
3. \( M(x; y; z; t) = M(y; z; x; t) \) (Symmetry about three variables)
4. \( M(x, y, z, t_1 + t_2 + t_3) \leq M(x, y, z, t_1) \cdot M(x, y, z, t_2) \cdot M(x, y, z, t_3) \) (This corresponds to tetrahedron inequality in 2-metric space) The function value \( M(x; y; z; t) \) may be interpreted as the probability that the area of triangle is less than \( t \):
5. \( M(x; y; z; \cdot) : [0; \infty) \rightarrow [0; 1] \) is left continuous.

Example 2.3: Let \((X; d)\) be a 2-metric space and denote \( a \star b = ab \) for all \( a; b \in [0; 1] \). For each \( h; m; n \in \mathbb{R}^+ \) and \( \forall t > 0 \); define \( M(x; y; z; t) = \frac{ht^m}{ht^m + md(x, y, z)} \). Then \((X; M; \star)\) is a fuzzy 2-metric space.

Example 2.4: Let \( X \) be the set \( \{1, 2, 3, 4\} \) with 2-metric \( d \) is defined by
\[
d(x, y, z)
= \begin{cases}
0, & \text{if } x = y, y = z, z = x \text{ and } \{x, y, z\} = \{1, 2, 3\} \\
\frac{1}{2}, & \text{otherwise}
\end{cases}
\]
for each \( t \in [0; \infty) \), define \( a \star b \star c = abc \) and
\[
M(x, y, z, t)
= \begin{cases}
0, & \text{if } t = 0 \\
\frac{t}{t + d(x, y, z)}, & \text{if } t > 0 \text{ where } x, y, z \in X
\end{cases}
\]
Then \((X; M; \star)\) is a fuzzy 2-metric space.

Definition 2.5: A sequence \( \{x_n\} \) in a fuzzy 2-metric space \((X; M; \star)\) is said to converge to \( x \in X \) if and only if \( \lim_{n \to \infty} M(x_n, x, a, t) = 1 \); for all \( a \in X \) and \( t > 0 \).

Definition 2.6: Let \((X; M; \star)\) be a fuzzy 2-metric space. A sequence \( \{x_n\} \) is called cauchy sequence if and only if \( \lim_{n \to \infty} M(x_{n+p}, x_n, a, t) = 1 \); for all \( a \in X \) and \( p > 0; t > 0 \).

Definition 2.7: A fuzzy 2-metric space \((X; M; \star)\) is said to be complete if and only if every cauchy sequence in \( X \) is convergent in \( X \).

Definition 2.8: Self mapping \( S \) and \( T \) of a fuzzy 2-metric space \((X; M; \star)\) are said to weakly commuting if \( M(STx; TSx; z; t) \geq M(Sx; Tx; t) \); for each \( x \in X \) and \( t > 0 \).

Definition 2.9: Self mapping \( S \) and \( T \) of a fuzzy 2-metric space \((X; M; \star)\) are said to be compatible if \( \lim_{n \to \infty} M(STx_n; TSx_n; z; t) = 1 \) \( \forall t > 0 \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( TX_n; Sx_n \to x \) for some \( x \in X \) as \( n \to \infty \).

**FIXED POINT THEOREMS IN FUZZY 2-METRIC SPACE**

Definition 2.10: Suppose \( S \) and \( T \) be self mappings of a fuzzy 2-metric space \((X; M; \star)\). A point \( x \) in \( X \) is called a coincidence point of \( S \) and \( T \) if and only if \( Sx = Tx \), then \( w = Sx = Tx \) is called a point of coincidence of \( S \) and \( T \).

Definition 2.11: Let \( X \) be a set, \( f; g \) are self maps of \( X \). A point \( x \) in \( X \) is called coincidence point of \( f \) and \( g \) if and only if \( fx = gx \). We shall call \( w = fx = gx \) a point of coincidence of \( f \) and \( g \).

Definition 2.12: Self maps \( S \) and \( T \) of a fuzzy 2-metric space \((X; M; \star)\) are said to be weakly compatible if they commute at their coincidence points that is \( Sx = Tx \) for some \( x \in X \) then \( STx = TSx \).

Definition 2.13: Self maps \( S \) and \( T \) of a fuzzy 2-
metric space \((X; M; \ast)\) are said to be occasionally weakly compatible (owc) if and only if there is a point \(x\) in \(X\) which is coincidence point of \(S\) and \(T\) at which they commute.

**Lemma 2.14:** Let \(X\) be a set, \(f; g\) owc self maps of \(X\). If \(f\) and \(g\) have a unique point of coincidence, \(w = fx = gx\); then \(w\) is the unique common fixed point of \(f\) and \(g\).

**IMPLICIT RELATION**

Let \(\Phi\) be the set of all real continuous function \(\phi : (R^+)^6 \rightarrow R^+\) satisfying the following condition \(\phi(u; u; v; v; u; u) \geq 0\) imply \(u \geq v\) for all \(u; v \in [0; 1]\):

**Theorem 3.1:** Let \((X; M; \ast)\) be a fuzzy 2-metric space with * continuous \(t\)-norm. Let \(A; B\) be two self mappings of \(X\) satisfying

1. The pair \((A; S)\) be owc.
2. For some \(\phi \in \Phi\) and for all \(x; y; z \in X\) and every \(t > 0\)
   \[
   \phi(M(Ax; Ay; z; t); M(Ax; Ay; z; t); M(Ax; Ax; z; t); M(Ax; Ay; z; t); M(Ax; Ay; z; t)) \geq 0.
   \]
then there exist a unique point \(w \in X\) such that \(Aw = Sw = w\). Moreover \(w\) is a unique common fixed point of \(A\) and \(S\).

**Proof:** Let the pair \((A; S)\) be owc. So there are points \(x; y; z \in X\) such that \(Ax = Ax\). We claim that \(Ax = Ay\). If not, by inequality (2),

\[
\phi(M(Ax; Ay; z; t); M(Ax; Ax; z; t); M(Ax; Ay; z; t); M(Ax; Ay; z; t)) \geq 0.
\]

In view of \(\phi\) we get \(Ax = Ay\). That is \(Ax = Sx = Ay = Sy\).

Suppose that there is another point \(w \in X\) such that \(Aw = Sw\) then by (1) we have \(Aw = Sw = By = Ty\). So \(Ax = Aw\) and \(w = Ax = Sx\) is the unique point of coincidence of \(A\) and \(S\): By lemma (2.14), \(w\) is a common fixed point of \(A\) and \(S\).

To prove the uniqueness: Let \(w_1; w_2\) be two common fixed points of \(A\) and \(S\). Assume that \(w_1 \neq w_2\).

**Theorem 3.2:** Let \((X; M; \ast)\) be a fuzzy 2-metric space with * continuous \(t\)-norm. Let \(A; B; S; T\) be four self mappings of \(X\) satisfying

1. The pairs \((A; S)\) and \((B; T)\) be owc.
2. For some \(\phi \in \Phi\) and for all \(x; y; z \in X\) and every \(t > 0\);
   \[
   \phi(M(Ax; By; z; t), M(Ax; By; z; t), M(Sx; Ty; z; t); M(Ax; By; z; t), M(Sx; By; z; t)) \geq 0.
   \]
then there exist a unique point \(w \in X\) such that \(Aw = Sw = w\) and a unique point \(z \in X\) such that \(Bz = Tz = z\). Moreover \(z = w\), so that there is a unique common fixed point of \(A; B; S\) and \(T\).

**Proof:** Let the pairs \((A; S)\) and \((B; T)\) be owc. So there are points \(x; y; z \in X\) such that \(Ax = Ax\) and \(By = Ty\). We claim that \(Ax = By\). If not, by inequality (2),

\[
\phi(M(Ax; By; z; t); M(Sx; Ty; z; t); M(Sx; Ax; z; t)); M(Sx; By; z; t)) \geq 0.
\]
\[ M(Ty; By; z; t); M(Ax; Ty; z; t); M(Sx; By; z; t) \geq 0 \]
\[ \phi \{ M(Ax; By; z; t); M(Ax; By; z; t); M(Ax; Ax; z; t); M(By; By; z; t); M(Ax; By; z; t); M(Ax; By; z; t) \} \geq 0 \]
\[ \phi \{ M(Ax; By; z; t); M(Ax; By; z; t); 1; 1; M(Ax; By; z; t); M(Ax; By; z; t) \} \geq 0 \]

In view of \( \Phi \) we get \( Ax = By \). That is \( Ax = Sx = By = Ty \).

Suppose that there is another point \( w \in X \) such that \( Aw = Sw \) then by (i) we have \( Ax = Sw = By = Ty \). So \( Ax = Aw \) and \( w = Ax = Sx \) is the unique point of coincidence of \( A \) and \( S \). By lemma (2.14) \( w \) is a common fixed point of \( A \) and \( S \).

And suppose that there is another point \( u \in X \) such that \( Bu = Tu \) then by (i) we have \( Ax = Sx = Bu = Tu \). So \( By = Bu \) and \( u = By = Ty \) is the unique point of coincidence of \( B \) and \( T \). By lemma (2.14) \( u \) is a common fixed point of \( B \) and \( T \).

Assume that \( w \neq u \) we have
\[ \phi \{ M(Aw; Bu; z; t); M(Sw; Tu; z; t); M(Sw; Aw; z; t) \}; \]
\[ M(Tu; Bu; z; t); M(Aw; Tu; z; t); M(Sw; Bu; z; t) \geq 0 \]
\[ \phi \{ M(w; u; z; t); M(w; u; z; t); M(w; w; z; t); M(u; u; z; t); M(w; u; z; t) \} \geq 0 \]
\[ \phi \{ M(w; u; z; t); M(w; u; z; t); 1; 1; M(w; u; z; t); M(w; u; z; t) \} \geq 0 \]

In view of \( \Phi \) we get \( w = u \). By lemma (2.14) \( z \) is a common fixed point of \( A; B; S \) and \( T \).

To prove the uniqueness:

Let \( w_1; w_2 \) be two common fixed points of \( A; B; S \) and \( T \).

Assume that \( w_1 \neq w_2 \).
\[ \phi \{ M(Aw_1; Bw_2; z; t); M(Sw_1; Tw_2; z; t); M(Sw_1; Aw_1; z; t); M(Tw_2; Bw_2; z; t); M(Aw_1; Tw_2; z; t); M(Sw_1; Bw_2; z; t) \} \geq 0 \]
\[ \phi \{ M(w_1; w_2; z; t); M(w_1; w_2; z; t); M(w_1; w_2; z; t); M(w_1; w_2; z; t) \} \geq 0 \]
\[ \phi \{ M(w_1; w_2; z; t); M(Aw_1; By; z; t); M( Ax; By; z; t); M(Ax; Ax; z; t); M(By; By; z; t); M(Ax; By; z; t) \} \geq 0 \]

Therefore we get \( w_1 = w_2 \).

**Theorem 3.3:** Let \( (X; M; *) \) be a fuzzy 2-metric space with \( \ast \) continuous tnorm. Let \( A; B; f; S; T; g \) be six self mappings of \( X \) satisfying
1. The pair \( (A; S); (B; T) \) and \( (f; g) \) be owc.
2. For some \( \phi \in \Phi \) and for all \( x; y; z \in X \) and every \( t > 0 \);
\[ \phi \{ M(Ax; By; fz; t); M(Sx; Ty; gz; t); M(Ax; Sx; fz; t); M(Ty; By; gz; t); M(Ax; Ty; fz; t); M(Sx; By; gz; t) \} \geq 0 \]
then there exist a unique point \( w \in X \) such that \( Aw = Sw = w \) and a unique point \( z \in X \) such that \( Bz = Tz = z \) and a unique point \( v \in X \) such that \( fv = gv = v \). Moreover \( w = z = v \); so that there is a unique common fixed point of \( A; B; f; S; T; g \).

**Proof:** Let the pairs \( \{A, S\}; \{B, T\} \) and \( \{f, g\} \) be owc.

Then there exist a unique point \( w \in X \) such that \( Aw = Sw = w \) and a unique point \( z \in X \) such that \( Bz = Tz = z \) and a unique point \( v \in X \) such that \( fv = gv = v \). Moreover \( w = z = v \); so that there is a unique common fixed point of \( A; B; f; S; T; g \).

We claim that \( Ax = By = fz = gz \).

So there are points \( x; y; z \in X \) such that \( Ax = Sx \) and \( By = Ty \) and \( fz = gz \).

We claim that \( Ax = By = fz \): If not, by inequality (2),
\[ \phi \{ M(Ax; By; fz; t); M(Sx; Ty; gz; t); M(Ax; Sx; fz; t); M(Ty; By; gz; t); M(Ax; Ty; fz; t); M(Sx; By; gz; t) \} \geq 0 \]
\[ \phi \{ M(Ax; By; fz; t); M(Ax; By; fz; t); M(Ax; Ax; fz; t); M(By; By; fz; t); M(Ax; By; fz; t); M(Ax; Ax; fz; t) \} \geq 0 \]
\[ M(By; By; fz; t); M(Ax; By; fz; t); M(Ax; By; fz; t) \geq 0. \]

\[ \phi \{ M(Ax; By; fz; t); M(Ax; By; fz; t); 1; 1; M(Ax; By; fz; t); M(Ax; By; fz; t) \} \geq 0. \]

In view of \( \Phi \) we get \( Ax = By = fz \). That is \( Ax = Sx = By = Ty = fz = gz \). Suppose that there is another point \( w \in X \) such that \( Aw = Sw \) then by (1) we have \( Aw = Sw = By = Ty = fz = gz \). So \( Ax = Aw \) and \( w = Ax = Sx \) is the unique point of coincidence of \( A \) and \( S \). By lemma(2.14) \( w \) is the only common fixed point of \( A \) and \( S \).

Suppose that there is another point \( u \in X \) such that \( Bu = Tu \) then by (i) we have \( Ax = Sx = Bu = Tu = fz = gz \). So \( By = Bu \) and \( u = By = Ty \) is the unique point of coincidence of \( B \) and \( T \). By lemma(2.14) \( u \) is the only common fixed point of \( B \) and \( T \). And suppose that there is another point \( v \in X \) such that \( fv = gv \) then by (i) we have \( Ax = Sx = By = Ty = fv = gv \). So \( fz = fv \) and \( v = fz = gz \) is the unique point of coincidence of \( f \) and \( g \). By lemma(2.14) \( v \) is the only common fixed point of \( f \) and \( g \). Assume that \( w \neq u \neq v \) we have

\[ \phi \{ M(Aw; Bu; fv; t); M(Sw; Tu; gv; t); M(Sw; Aw; fv; t); M(Tu; Bu; gv; t); M(Aw; Tu; fv; t); M(Sw; Bu; gv; t) \} \geq 0 \]

\[ \phi \{ M(w; w; v; t); M(w; v; w; v; t); M(w; w; w; v; t); M(w; u; v; t); M(w; v; u; v; t); M(w; u; v; t); M(w; v; u; t); M(w; v; t) \} \geq 0 \]

In view of \( \Phi \) we get \( w = u = v \). By lemma(2.14) \( w \) is a common fixed point of \( A; B; f; S; T \) and \( g \).

To prove the uniqueness: Let \( w_1, w_2 \) be two common fixed points of \( A; B; f; S; T \) and \( g \). Assume that \( w_1 \neq w_2 \):

\[ \phi \{ M(Aw_1; Bw_2; v; t); M(Sw_1; Tw_2; v; t); M(Sw_1; Aw_1; v; t); M(Tw_2; Bw_2; v; t); M(Aw_1; Tw_2; v; t); M(Sw_1; Bw_2; v; t) \} \geq 0 \]

\[ \phi \{ M(w_1; w_2; v; t); M(w_1; w_2; v; t); 1; 1; M(w_1; w_2; v; t); M(w_1; w_2; v; t) \} \geq 0. \]

Therefore we get \( w_1 = w_2 \).

REFERENCES


